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(Presented by Academician L.S. Pontryagin on 8 January 1959)

In the theory of automatic control, ever greater importance is being attached to the problem of creating systems optimal from the standpoint of a given criterion (time of response, efficiency, power expenditure). L.S. Pontryagin's Maximum Principle, formulated (ref. 1) and proved (2) at first for the case of time optimization, and then generalized (3) for optimum response in general, defines the necessary optimum conditions for a broad class of variational problems arising in the theory of optimal systems. The present paper, in many respects relying upon ideas put forth in (1) and developed in (2 and 3), established certain sufficient conditions for optimality in the case of a single variational problem, to which may be reduced a number of automatic control problems (specifically, variations of B.V. Bulgakov's problem (4,5) on disturbance build-up in dynamic systems).

1. Statement of the problem. Let us examine the movement of a point in an n-dimensional phase space $x = (x_1, \ldots, x_n)$ described by the system of differential equations

$$\dot{x} = f(x, u, t), \qquad (1)$$

where $f = (f_1, \ldots, f_n)$, $u = (u_1, \ldots, u_r)$. Let us term the variable vector $u(t) = (u_1(t), \ldots, u_r(t))$ the "control" (1).

As a class of permissible controls, let us take a class of piecewise continuous vector functions $u(t)$ varying within a certain fixed closed set U of an r-dimensional space $R(u_1,\ldots,u_r)$. Let us consider the functions $f_i (i = 1,\ldots,n)$ continuous for the combination of arguments (x,u,t) and as having continuous partial derivatives with respect to the arguments (x,u) through the second order inclusively.

We now state the following problem: with a fixed initial position of the point $x(T_0) = x^0$ to choose a control $u(t) \in U$ such that the sum $S \equiv \sum_1^n c_i x_i(T)$ at a given moment of $t = T$ takes on a minimum (maximum) value.

The control which yields the minimum (maximum) value for the functional S will be termed min-optimal (max-optimal) in S.

2. <u>Increments of the functional with variations in the control.</u> Let us introduce a vector $p(t) = (p_1(t),\ldots,p_n(t))$ and a function $H(x,p,u,t)$ bound by the conditions (1):

$$H \equiv \sum_1^n p_s f_s(x,u,t), \qquad (2)$$

$$\dot{p}_i = -\sum_1^n p_s \frac{\partial f_s}{\partial x_i}, \quad i = 1,\ldots,n. \qquad (3)$$

Let us write down the system (1), (3) with the aid of (2) in the following form (ref. 1):

$$\dot{x}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial x_i}, \quad i = 1,\ldots,n. \qquad (4)$$

The vector $p(t)$ is uniquely determined by (4), provided that $u(t)$ and the corresponding boundary conditions are specified.

The combination of (2) and (4) yields the equation

$$I(x,p,u) \equiv \int_{T_0}^{T} \Big[\sum_1^n p_i \dot{x}_i - H(x,p,u,t)\Big] dt = 0 \qquad (5)$$

which holds true if (4) is satisfied for arbitrary controls and boundary conditions. Let us choose some control $u(t) \in U$ and examine its increment $\delta u(t)$. Let $x(t), p(t)$ be the solution of system (4) for the control $u(t)$ and certain boundary conditions, while $x(t) + \delta x(t)$, $p(t) + \delta p(t)$ is the solution of system (4) for the control $u(t) + \delta u(t)$ with the same boundary conditions. Let us consider the difference $\Delta \equiv I(x+\delta x, p+\delta p, u+\delta u) - I(x,p,u)$, with $\Delta = 0$ at all times. The corresponding development using (4) leads to

the following expression for Δ:

$$\Delta = \sum_1^n [p_i(t) + \tfrac{1}{2}\delta p_i(t)]\delta x_i(t)\Big|_{T_0}^{T} -$$

$$- \int_{T_0}^{T} [H(x, p, u+\delta u, t) - H(x, p, u, t)]\,dt - \eta. \quad (6)$$

Here $\eta = \eta_1 + \eta_2$ and

$$\eta_1 = \frac{1}{2}\int_{T_0}^{T}\sum_1^{2n}\left[\frac{\partial H(y, u+\delta u, t)}{\partial y_s} - \frac{\partial H(y, u, t)}{\partial y_s \partial y_q}\right]\delta y_s dt, \quad (7)$$

$$\eta_2 = \frac{1}{2}\int_{T_0}^{T}\sum_{q,s=1}^{2n}\left[\frac{\partial^2 H(y+\theta_1\delta y, u+\delta u, t)}{\partial y_s \partial y_q} - \frac{\partial^2 H(y+\theta_2\delta y, u+\delta u, t)}{\partial y_s \partial y_q}\right]\delta y_s \delta y_q dt$$

where $0<\theta_1<1$, $0<\theta_2<1$ and the vector

$$y = (y_1, \dots, y_{2n})(x_i = y_i,\ p_i = y_{n+i},\ i = 1, \dots, n)$$

introduced for the sake of brevity.

Let us set the following expressions as the boundary conditions for (4)

$$x_i(T_0) = x_i^0\ ;\ p_i(T) = -c_i,\ i = 1, \dots, n, \quad (8)$$

requiring thereby that $\delta x_i(T_0) = \delta p_i(T) = 0$. Then, taking into account that $\Delta = 0$, we obtain from (6) that

$$\delta S = \sum_1^n c_i \delta x_i(T) = -\int_{T_0}^{T}[H(x, p, u+\delta u, t) - H(x, p, u, t)]dt - \eta \quad (9)$$

Let us evaluate the remainded term in (9). Let the increment $\delta u(t)$ differ from zero only on the segment $[t_1, t_2] \in [T_0, T](t_2 - t_1 = \tau)$. There is the following expression for the increment of the solution of system (4) for boundary conditions of the type (8):

$$|\delta y_s(t)| \le M \int_{t_1}^{t_2}\sum_1^r |\delta u_k(z)|\,dz\ ,\ s = 1, \dots, 2n,\ t \in [T_0, T], \quad (10)$$

where M is a constant independent of t_1, t_2, and $\delta u(x)$.

From (10) it easily follows for η that

$$|\eta| \le C\tau \int_{t_1}^{t_2}\sum_1^r \delta u_k^2(t)\,dt \quad (11)$$

where C is a constant independent of t_1, t_2, and $\delta u(t)$.

3. <u>Definitions.</u> Let $[T_1, T_2]$ be a segment lying in $[T_0, T]$ (or, possibly, coinciding with the latter). We will call the control $u(t) \in U$ min-optimal (max-optimal) in S:

1) in the small segment $[T_1, T_2]$, if there exists a sufficiently small number ε, such that the functional S reaches a least (greatest) value in u(t) of all values on all three controls $u(t) + \delta u(t) \in U$,

fro which $\max\limits_{t \in [T_1, T_2]} \sum\limits_{1}^{r} |\delta u_k(t)| \leq \varepsilon$ and $\delta u(t) = 0$ are external to the segment $[T_1, T_2]$;

2) in small portions of the segment $[T_1, T_2]$, provided there exists a sufficiently small number such that the functional S reaches a least (greatest) value on u(t) of all values on all three controls $u(t) + \delta u(t) \in U$ for which $\delta u(t) = 0$ externally to the segment $[t_1, t_2] \in [T_1, T_2]$ which is arbitrary, but such that $t_2 - t_1 \leq \tau$;

3) for a small magnitude in the small segment $[T_1, T_2]$, if there exists such a pair of sufficiently small numbers ε, τ for which the functional S reaches a least (greatest) value on u(t) of all values on all three controls $u(t) + \delta u(t) \in U$ for which $\delta u(t) = 0$ externally to the segment $[t_1, t_2] \in [T_1, T_2]$ which is arbitrary, but such that $t_2 - t_1 \leq \tau$, and, in addition, $\max\limits_{t \in [t_1, t_2]} \sum\limits_{1}^{r} |\delta u_k(t)| \leq \varepsilon$.

4. <u>The maximum condition.</u> Let $u(t) \in U$ be some control determined on the interval $[T_0, T]$, and x(t), p(t) -- the solution of the system (4) with boundary conditions (8) for the control u(t). We will say that control u(t) satisfies the maximum (minimum) condition on $[T_0, T]$ if the function $G(u,t) \equiv H(x(t), p(t), u, t)$ reaches and absolute maximum (minimum) along $u = (u_1, \ldots, u_r)$ on the set U for $u = u(t)$ and arbitrary $t \in [T_0, T]$.

The necessity of the maximum (minimum) condition for the min- (max-) optimality of the control u(t) is easily proved on the basis of formulas (9) and (11) (considering the increments $\delta u(t)$ with sufficiently small τ) [See Note] [Note: The necessity of the maximum (minimum) condition also follows easily from L.S. Pontryagin's Maximum Principle].

5. <u>Theorem I.</u> Let the control u(t) satisfy the maximum (minimum) condition on the segment $[T_0, T]$; furthermore,

let $[T_1, T_2]$ be a segment lying within $[T_0, T]$ (or, possibly, coinciding with it).

Then,

1) if there exists a constant $A > 0$ such that the inequality

$$|H(x(t), p(t), u(t) + v, t) - H(x(t), p(t) u(t), t)| > A \sum_{1}^{r} |v_k|$$

is satisfied for any $t \in [T_1, T_2]$ and arbitrary, sufficiently small $v = (v_1, \ldots, v_r)$, $u + v \in U$, then the control $u(t)$ is min-optimal (max-optimal) on S in a small magnitude within the segment $[T_1, T_2]$.

2) if there exists a constant $B > 0$ such that the inequality

$$|H(x(t), p(t), u(t) + v, t) - H(x(t), p(t), u(t), t)| > B \sum_{1}^{r} v_k^2$$

is satisfied for any $t \in [T_1, T_2]$ and arbitrary sufficiently small v, $u(t) + v \in U$, the control $u(t)$ is min- (max-) optimal on S in small portions of the segment $[T_1, T_2]$.

3) if the conditions of point 2) of the present theorem are satisfied, the set U is a compact one, and in addition,

$$H(x(t), p(t), u(t) + v, t) - H(x(t), u(t), t) \neq 0$$

for any $v \neq 0$, $u(t) + v \in U$, then the control $u(t)$ is min- (max-) optimal on S on small portions of the segment $[T_1, T_2]$.

The proof of the theorem follows easily from (9) and (11).

6. Linear systems. Let us consider a class of problems for which the system (1) is linear in x:

$$\dot{x}_i = \sum_{1}^{n} a_{is}(t) x_s + \varphi_i(u_1, \ldots, u_r, t) \qquad i = 1, \ldots, n. \tag{12}$$

Theorem II. A necessary and sufficient condition for the min- (max-) optimality of control $u(t)$ on $S = \sum_{1}^{n} c_i x_i(T)$ for the system (12) is the fulfillment of the maximum (minimum) condition.

The proof of its sufficiency follows out of the fact that $\eta = 0$ in (9). The latter is easily shown by bearing in mind that $\partial^2 H / \partial x_q \partial x_s \equiv 0$, $\partial^2 H / \partial p_q \partial p_s \equiv 0$, $\partial^2 H / \partial p_q \partial x_s \equiv a_{qs}(t)$, and finally that $\delta p_q(t) \equiv 0$ $(q, s = 1, \ldots, n)$.

The necessity for the maximum (minimum) condition has already been noted in Section 4.

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